NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N.Y.

IMM-NYU 242
SEPTEMBER 1957

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

IMM-242

Perturbation of Continuous Spectra and Singular Integral Operators

W. KOPPELMAN *and* J. PINCUS

PREPARED UNDER
CONTRACT No. DA-30-069-ORD-1258
WITH THE
OFFICE OF ORDNANCE RESEARCH
UNITED STATES ARMY

REPRODUCTION IN WHOLE OR IN PART
IS PERMITTED FOR ANY PURPOSE
OF THE UNITED STATES GOVERNMENT.

IMM-NYU 242
c.1

IMM-NYU 242
September 1957

New York University
Institute of Mathematical Sciences

PERTURBATION OF CONTINUOUS SPECTRA AND SINGULAR INTEGRAL OPERATORS

by

W. Koppelman and J. Pincus

Prepared under the auspices of Contract DA-30-069-ORD-1258 with the Office of Ordnance Research, United States Army.

New York, 1957

Acknowledgement

The authors are indebted to Professor K. O. Friedrichs for his valuable suggestions.

1. Introduction

This is a report on the study of a certain problem associated with the perturbation of continuous spectra, where the perturbing operator is a singular integral operator. Here we shall consider only the special one-parameter family of operators

$$L_\varepsilon x(\lambda) = \lambda x(\lambda) + \frac{\varepsilon}{\pi i} P \int_a^b \frac{x(\mu)}{\mu-\lambda} d\mu \tag{1.1}$$

whose domains consist of the space $L^2(a,b)$ of square integrable functions $x(\lambda)$ defined on the interval (a,b) of the real axis of the complex plane. (The integral which occurs in (1.1) is to be interpreted as a Cauchy principal value.) For convenience we shall assume that $\varepsilon > 0$.

The investigation of (1.1) was initiated in order to provide an example for the following problem:

Let L be the operator L_0 of the expression (1.1) . This operator is symmetric and possesses a simple continuous spectrum which consists of the closed interval from a to b . Find bounded symmetric operators K for which the perturbation $L + \varepsilon K$ of the operator L causes a simple shift of the spectrum in a nontrivial manner.[1]

A general estimate of the spectral shift for the perturbation $L + \varepsilon K$ may be easily derived. We state this estimate in the form of

Lemma 1.1:

The spectrum of the operator $L + \varepsilon K$ lies within the closed interval from $a - \varepsilon||K||$ to $b + \varepsilon||K||$, where $||K||$ stands for the norm of K.

Proof: Let λ_0 be any point of the real axis which does not belong to this interval, so that the distance of λ_0 from this interval is a positive number δ. Suppose that λ_0 belongs to the spectrum of L_ε. Then there exists a sequence $\{x_n\}$, $x_n \varepsilon L^2(a,b)$, $||x_n|| = 1$, such that

$$(1.2) \qquad \lim_{n\to\infty} ||\{L + \varepsilon K\}x_n - \lambda_0 x_n|| = 0 .^2$$

Let δ_0 be an arbitrary positive number ; then,

$$||(L - \lambda_0)x_n - \varepsilon K x_n|| < \delta_0 \ , \ n > N(\delta_0)$$

so that

$$||(L - \lambda_0)x_n|| < \delta_0 + \varepsilon||K|| \ .$$

On the other hand,

$$||(L - \lambda_0)x_n|| \geq \delta + \varepsilon||K|| \ ,$$

since the distance of λ_0 from the interval $(a - \varepsilon||K||, b + \varepsilon||K||$ was δ. Thus we have $\delta < \delta_0$. Since δ is fixed while δ_0 is arbitrary, we have a contradiction.

In Section 4, we shall prove that in our case $||K|| = 1$. From the spectral representation of the operator (1.1) which we derive in Section 6, we shall be able to conclude that L_ε has a simple continuous spectrum which consists of the closed interval from $a - \varepsilon$ to $b + \varepsilon$. Thus the shift produced by the perturbation (1.1) is essentially maximal.

The spectral analysis of (1.1) is based upon an attempt to obtain an eigenfunction expansion for the operator L_ε, i.e. an isometric transformation of $L^2(a-\varepsilon, b+\varepsilon)$ onto $L^2(a,b)$ which is of the form $\int_{a-\varepsilon}^{b+\varepsilon} G(\xi,\lambda,\varepsilon)g(\xi)d\xi$, $a < \lambda < b$, where $G(\xi,\lambda,\varepsilon)$ is an improper eigenfunction of the operator L_ε, corresponding to the improper eigenvalue ξ. Naturally, the work here leans heavily upon the theory of singular integral equations. A clear exposition of this theory may be found in the works of N. I. Muskhelishvili [3] and S. G. Michlin [2].

2. Improper Eigenfunctions.

As usual, a function $x(\lambda)$ is called an eigenfunction of the operator L_ε corresponding to the eigenvalue ℓ, if

$$L_\varepsilon x(\lambda) = \ell x(\lambda) . \tag{2.1}$$

In order to find such functions $x(\lambda)$, we must examine the singular integral equation

$$(\lambda - \ell)x(\lambda) + \frac{\varepsilon}{\pi i} P \int_a^b \frac{x(\mu)}{\mu-\lambda} d\mu = 0 . \tag{2.2}$$

We now suppose that $x(\lambda)$ is a solution of (2.2), which satisfies the following conditions:

(i) On any closed subinterval (a', b') of the interval (a, b), which does not include the end points a, b, the function $x(\lambda)$ satisfies a Hölder condition;

(ii) near the ends a, b, the function $x(\lambda)$ is of the form

$$x(\lambda) = \frac{x^*(\lambda)}{(\lambda - c)^\gamma} , \quad \gamma = \alpha_0 + i\beta_0 , \quad 0 \leq \alpha_0 < 1 ,$$

where c is either of the ends a, b, α_0 and β_0 are real constants and $x^*(\lambda)$ satisfies a Hölder condition near and at c.

The analytic function $X(z)$ which is defined by the relation

$$X(z) = \frac{1}{2\pi i} \int_a^b \frac{x(\mu)}{\mu - z} \, d\mu \tag{2.3}$$

then satisfies the barrier equation

$$(\lambda - \ell + \varepsilon) X^+(\lambda) - (\lambda - \ell - \varepsilon) X^-(\lambda) = 0 , \quad a < \lambda < b , \tag{2.4}$$

where $X^+(\lambda) = \lim\limits_{z \to \lambda, \, \mathrm{Im}\, z > 0} X(z)$ and $X^-(\lambda) = \lim\limits_{z \to \lambda, \, \mathrm{Im}\, z < 0} X(z)$, since $X^+(\lambda) + X^-(\lambda) = \frac{1}{\pi i} P \int_a^b \frac{x(\mu)}{\mu - \lambda} \, d\mu$ and $X^+(\lambda) - X^-(\lambda) = x(\lambda)$. Furthermore

$$X(\infty) = 0 , \tag{2.5}$$

$$X(z) = \mathcal{O}\left(\frac{1}{(z-a)^{\alpha}}\right) , \quad \alpha < 1 \text{ , near } a \tag{2.6}$$

and

$$X(z) = \mathcal{O}\left(\frac{1}{(z-b)^{\beta}}\right) , \quad \beta < 1 \text{ , near } b \text{ . } [3] \text{ .} \tag{2.7}$$

The fundamental solution of equation

(2.4) is given by the expression

$$E(\ell, z, \varepsilon) = e^{\frac{1}{2\pi i}\int_a^b \log \frac{\mu-\ell-\varepsilon}{\mu-\ell+\varepsilon}\frac{d\mu}{\mu-z}} \text{ .} \tag{2.8}$$

It can be shown that $E(\ell, z, \varepsilon)$ satisfies the conditions (2.6), (2.7), while $E(\ell, \infty, \varepsilon) = 1$.

If $X(z)$ is a solution of (2.4), then

$$\frac{X^{+}(\lambda)}{E^{+}(\ell, \lambda, \varepsilon)} - \frac{X^{-}(\lambda)}{E^{-}(\ell, \lambda, \varepsilon)} = 0 \text{ ,} \tag{2.9}$$

so that if $X^{+}(\lambda)$, $X^{-}(\lambda)$ are continuous for $a < \lambda < b$,

$$X(z) = E(\ell, z, \varepsilon)P(z) \text{ ,} \tag{2.10}$$

where $P(z)$ is regular in the entire plane, except, perhaps, at the points a and b . If $X(\infty) = 0$, then we must have $P(\infty) = 0$.

If $(\lambda-\ell)^2 - \varepsilon^2 \neq 0$, this process may furnish a solution $X(z)$ of (2.4) which also satisfies (2.5), (2.6) and (2.7). In that case, $x(\lambda) = X^+(\lambda) - X^-(\lambda)$ is an integrable function which satisfies (2.2). For real values of ℓ which lie outside the interval $(a-\varepsilon, b+\varepsilon)$, the problem (2.4) can never have a non-trivial solution which also satisfies (2.5), (2.6), and (2.7), since in this case $E(\ell,z,\varepsilon)$ is bounded near the end points a,b and does not vanish there.

The condition $(\lambda-\ell)^2 - \varepsilon^2 \neq 0$ was needed to insure the integrability of the function $x(\lambda)$. We now relinquish this integrability. Any function $x(\lambda) = X^+(\lambda) - X^-(\lambda)$, where $X(z)$ satisfies (2.4), (2.5), (2.6) and (2.7), will be called an improper eigenfunction of the operator L_ε corresponding to the eigenvalue ℓ.

If we set

$$E_+(\xi,z,\varepsilon) = \lim_{\ell \to \xi,\, \mathrm{Im}\, \ell > 0} E(\ell,z,\varepsilon) \quad ,$$

$$E_-(\xi,z,\varepsilon) = \lim_{\ell \to \xi,\, \mathrm{Im}\, \ell < 0} E(\ell,z,\varepsilon) \quad ,$$

where ξ is real, then the function

$$F(\xi,z,\varepsilon) = E_+(\xi,z,\varepsilon) - E_-(\xi,z,\varepsilon) \tag{2.11}$$

furnishes the improper eigenfunction

$$G(\xi,\lambda,\varepsilon) = F^{+}(\xi,\lambda,\varepsilon) - F^{-}(\xi,\lambda,\varepsilon) .$$

(Observe that if $\xi < a - \varepsilon$ or $\xi > b + \varepsilon$, then $F(\xi,z,\varepsilon) = 0$.)

3. Relations for the Fundamental Solution.

The function $E(\ell,z,\varepsilon)$, of course, satisfies

$$(3.1) \qquad (\lambda-\ell+\varepsilon)E^{+}(\ell,\lambda,\varepsilon) - (\lambda-\ell-\varepsilon)E^{-}(\ell,\lambda,\varepsilon) = 0 , \qquad a < \lambda < b .$$

By analyzing $E(\ell,z,\varepsilon)$ directly, one may also establish that

$$(3.2) \qquad \{\min(\xi+\varepsilon,b)-z\}\, E_{+}(\xi,z,\varepsilon) - \{\max(\xi-\varepsilon,a)-z\}\, E_{-}(\xi,z,\varepsilon) = 0 , \qquad a-\varepsilon < \xi < b+\varepsilon .$$

It is interesting to observe that the fundamental solution

$$D(z,\ell,\varepsilon) = e^{\frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \log \frac{\max(\nu-\varepsilon,a)-z}{\min(\nu+\varepsilon,b)-z}\frac{d\nu}{\nu-\ell}}$$

of the barrier equation (3.2) is equal to $E(\ell,z,\varepsilon)$. This may be shown through integration by parts. Furthermore,

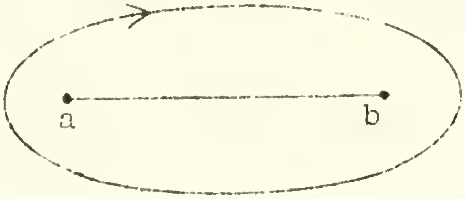
$$(3.3) \qquad E(\ell,z,\varepsilon) = \overline{E(\overline{\ell},\overline{z},-\varepsilon)} = \frac{1}{E(\ell,z,-\varepsilon)} .$$

We now perform several calculations which lead to results which are fundamental to our later investigations. We first remark that $E(\ell,z,\varepsilon)$ satisfies conditions (2.6) and (2.7) with $\alpha,\beta < \frac{1}{2}$.

Next, we compute the integral

$$\frac{1}{2\pi i}\oint E(\ell_1,z,\varepsilon)E(\ell_2,z,-\varepsilon)dz$$

over the contour described in the diagram below:



By the Cauchy residue theorem, we can conclude that

$$(3.4)\quad \frac{1}{2\pi i}\oint E(\ell_1,z,\varepsilon)E(\ell_2,z,-\varepsilon)dz = \frac{1}{2\pi i}\oint_a^b \log\frac{\mu-\ell_1-\varepsilon}{\mu-\bar{\ell}_1+\varepsilon}\,d\mu - \frac{1}{2\pi i}\int_a^b \log\frac{\mu-\ell_2-\varepsilon}{\mu-\bar{\ell}_2+\varepsilon}\,d\mu \ .$$

On the other hand, we also have

$$(3.5)\quad \frac{1}{2\pi i}\oint E(\ell_1,z,\varepsilon)E(\ell_2,z,-\varepsilon)dz = \frac{1}{2\pi i}\oint E(\ell_1,z,\varepsilon)E(\bar{\ell}_2,\bar{z},\varepsilon)dz \text{ (from (3.}$$

$$= \frac{1}{2\pi i}\int_a^b \{E^+(\ell_1,\lambda,\varepsilon)\overline{E^-(\ell_2,\lambda,\varepsilon)} - E^-(\ell_1,\lambda,\varepsilon)\overline{E^+(\bar{\ell}_2,\lambda,\varepsilon)}\}d\lambda \ .$$

We now define

$$(3.6)\qquad H(\ell,\lambda,\varepsilon) = E^+(\ell,\lambda,\varepsilon) - E^-(\ell,\lambda,\varepsilon) \ .$$

Through Relation (3.1), we can prove

$$(3.7)\quad (\ell_1-\ell_2)\{\overline{E^+(\bar{\ell}_2,\lambda,\varepsilon)}-\overline{E^-(\bar{\ell}_2,\lambda,\varepsilon)}\}\{E^+(\ell_1,\lambda,\varepsilon)-E^-(\ell_1,\lambda,\varepsilon)\}$$

$$= (\ell_1-\ell_2)\overline{H(\bar{\ell}_2,\lambda,\varepsilon)}H(\ell_1,\lambda,\varepsilon)$$

$$= 2\varepsilon\{\overline{E^+(\bar{\ell}_2,\lambda,\varepsilon)}E^-(\ell_1,\lambda,\varepsilon) - \overline{E^-(\bar{\ell}_2,\lambda,\varepsilon)}E^+(\ell_1,\lambda,\varepsilon)\} .$$

Thus, from (3.4), (3.5), and (3.7), we have

$$(3.8)\quad \frac{\ell_1-\ell_2}{2\pi i}\int_a^b \overline{H(\bar{\ell}_2,\lambda,\varepsilon)}H(\ell_1,\lambda,\varepsilon)\,d\lambda = \frac{\varepsilon}{\pi i}\{\int_a^b \log\frac{\mu-\ell_2-\varepsilon}{\mu-\ell_2+\varepsilon}\,d\mu - \int_a^b \log\frac{\mu-\ell_1-\varepsilon}{\mu-\ell_1+\varepsilon}\,d\mu\} .$$

By integration by parts, we may show that

$$(3.9)\quad \int_a^b \log\frac{\mu-\ell-\varepsilon}{\mu-\ell+\varepsilon}\,d\mu = \int_{a-\varepsilon}^{b+\varepsilon}\frac{\max(\nu-\varepsilon,a)-\min(\nu+\varepsilon,b)}{\nu-\ell}\,d\nu ,$$

so that

$$(3.10)\quad \int_a^b \overline{H(\bar{\ell}_2,\lambda,\varepsilon)}H(\ell_1,\lambda,\varepsilon)\,d\lambda$$

$$= \frac{2\varepsilon}{\ell_1-\ell_2}\int_{a-\varepsilon}^{b+\varepsilon}\{\max(\nu-\varepsilon,a)-\min(\nu+\varepsilon,b)\}\{\frac{1}{\nu-\ell_2}-\frac{1}{\nu-\ell_1}\}d\nu .$$

If γ is real, with $a-\varepsilon < \gamma < b+\varepsilon$, and $\eta > 0$, then from (3.10) we immediately obtain

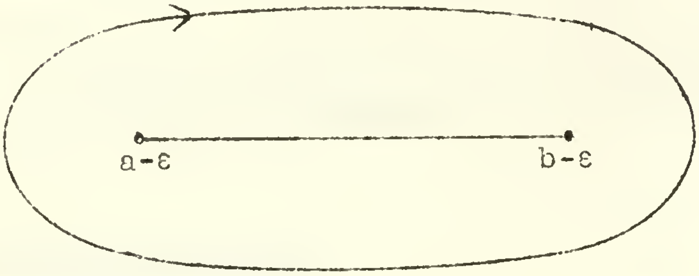
$$(3.11)\quad \lim_{\eta\to 0} \frac{1}{2\pi i}\int_a^b \overline{H(\bar{\ell}_2,\lambda,\varepsilon)}\{H(\gamma+i\eta,\lambda,\varepsilon) - H(\gamma-i\eta,\lambda,\varepsilon)\}d\lambda$$

$$= -\frac{2\varepsilon}{\gamma-\ell_2}\{\max(\gamma-\varepsilon,a) - \min(\gamma+\varepsilon,b)\} .$$

We now remark that $E(\ell,z,\varepsilon)$ is bounded for ℓ near the points $a-\varepsilon$, $b+\varepsilon$. Next, we evaluate the integral

$$\frac{1}{2\pi i}\oint E(\ell,z_1,\varepsilon)E(\ell,z_2,-\varepsilon)d\ell$$

over the contour described in the diagram below:



By the Cauchy residue theorem we can conclude that

$$(3.12)\quad \frac{1}{2\pi i}\oint E(\ell,z_1,\varepsilon)E(\ell,z_2,-\varepsilon)d\ell = -\frac{\varepsilon}{\pi i}\int_a^b \frac{d\mu}{\mu-z_1} + \frac{\varepsilon}{\pi i}\int_a^b \frac{d\mu}{\mu-z_2} .$$

Furthermore,

$$(3.13)\quad \frac{1}{2\pi i}\oint E(\ell,z_1,\varepsilon)E(\ell,z_2,-\varepsilon)d\ell = \frac{1}{2\pi i}\oint E(\ell,z_1,\varepsilon)\overline{E(\bar{\ell},\bar{z}_2,\varepsilon)}d\ell$$

$$= \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \{E_+(\xi,z_1,\varepsilon)\overline{E_-(\xi,\bar{z}_2,\varepsilon)} - E_-(\xi,z_1,\varepsilon)\overline{E_+(\xi,\bar{z}_2,\varepsilon)}\}d\xi .$$

From (3.2) we can show that

$$(3.14)\quad (z_1-z_2)F(\xi,z_1,\varepsilon)\overline{F(\xi,\overline{z}_2,\varepsilon)}$$

$$= \{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)\}\{E_+(\xi,z_1,\varepsilon)\overline{E_-(\xi,\overline{z}_2,\varepsilon)} - E_-(\xi,z_1,\varepsilon)\overline{E_+(\xi,\overline{z}_2,\varepsilon)}\}$$

Thus

$$(3.15)\quad \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,z_1,\varepsilon)\overline{F(\xi,\overline{z}_2,\varepsilon)}}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)}\,d\xi = \frac{\varepsilon}{\pi i(z_2-z_1)}\left\{\int_a^b \frac{d\mu}{\mu-z_1} - \int_a^b \frac{d\mu}{\mu-z_2}\right\}.$$

Furthermore, for real ω , $a < \omega < b$, and $\eta > 0$

$$(3.16)\quad \lim_{\eta\to 0} \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,z_1,\varepsilon)\{\overline{F(\xi,\omega+i\eta,\varepsilon)} - \overline{F(\xi,\omega-i\eta,\varepsilon)}\}d\xi}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)} = \frac{2\varepsilon}{\omega-z_1} \quad .$$

4. The Finite Hilbert Transform.

We interrupt the continuity of the discussion now, in order to investigate our choice of K , the operator

$$(4.1)\qquad Kx(\lambda) = \frac{1}{\pi i}\,P\int_a^b \frac{x(\mu)}{\mu-\lambda}\,d\mu \ .$$

The eigenfunctions associated with this operator are found by examining the singular integral equation

$$(4.2)\qquad \lambda' x(\lambda) - \frac{1}{\pi i}\,P\int_a^b \frac{x(\mu)}{\mu-\lambda}\,d\mu = 0 \quad .$$

We now proceed as in Section 2. If $x(\lambda)$ is a solution of (4.2) with properties (i) and (ii) of Section 2, then the analytic function

$$X(z) = \frac{1}{2\pi i} \int_a^b \frac{x(\mu)}{\mu - z}\, d\mu \tag{4.3}$$

satisfies the barrier equation

$$(\ell - 1)X^{+}(\lambda) - (\ell + 1)X^{-}(\lambda) = 0 , \quad a < \lambda < b , \tag{4.4}$$

and the conditions (2.5), (2.6) and (2.7) . Conversely, if $\ell \neq \pm 1$, every solution $X(z)$ of (4.4), which also satisfies (2.5), (2.6) and (2.7), furnishes a solution $x(\lambda)$ of (4.2) through the equation $x(\lambda) = X^{+}(\lambda) - X^{-}(\lambda)$.

The fundamental solution $\mathcal{E}(\ell, z)$ of (4.4) has the form

$$\mathcal{E}(\ell, z) = e^{\frac{1}{2\pi i}(\log \frac{\ell+1}{\ell-1}) \int_a^b \frac{d\mu}{\mu - z}} . \tag{4.5}$$

We now restrict ourselves to the values $\ell = \xi$, where ξ is real and not equal to 1 or -1 . Clearly, a solution of (4.4) which also satisfies (2.5), (2.6) and (2.7) is given by

$$\mathcal{F}(\xi, z) = \mathcal{E}_{+}(\xi, z) - \mathcal{E}_{-}(\xi, z) , \tag{4.6}$$

where

$$\mathcal{E}_{+}(\xi, z) = \lim_{\ell \to \xi,\, \mathrm{Im}\, \ell > 0} \mathcal{E}(\ell, z) ,$$

$$\mathcal{E}_{-}(\xi, z) = \lim_{\ell \to \xi,\, \mathrm{Im}\, \ell < 0} \mathcal{E}(\ell, z) .$$

✓3

(Observe that $\mathcal{F}(\xi,z) = 0$ for $|\xi| > 1$.) It can furthermore be shown that $\mathcal{F}(\xi,z)$ is the only linearly independent solution of (4.4) which also satisfies the subsidiary conditions.

Of course when $|\xi| < 1$,

$$\mathcal{E}_+(\xi,z) = e^{\frac{1}{2\pi i}\left(\log\frac{1+\xi}{1-\xi} - \pi i\right)\int_a^b \frac{d\mu}{\mu - z}} ,$$

$$\mathcal{E}_-(\xi,z) = e^{\frac{1}{2\pi i}\left(\log\frac{1+\xi}{1-\xi} + \pi i\right)\int_a^b \frac{d\mu}{\mu - z}} ,$$

so that

$$(4.7) \qquad \mathcal{F}(\xi,z) = e^{-\frac{1}{2\pi}\left(\log\frac{1+\xi}{1-\xi}\right)\overset{\log\left|\frac{b-a}{a-z}\right|}{\wedge}} \left\{\sqrt{\frac{a-z}{b-z}} - \sqrt{\frac{b-z}{a-z}}\right\} .$$

We obtain an improper eigenfunction for the finite Hilbert transform, by forming

$$\mathcal{F}^+(\xi,\lambda) - \mathcal{F}^-(\xi,\lambda) , \qquad a < \lambda < b ,$$

where

$$\mathcal{F}^+(\xi,\lambda) = \lim_{z\to\lambda,\, \operatorname{Im} z > 0} \mathcal{F}(\xi,z)$$

$$\mathcal{F}^-(\xi,\lambda) = \lim_{z\to\lambda,\, \operatorname{Im} z < 0} \mathcal{F}(\xi,z) .$$

An easy computation shows that

$$(4.8)\quad \mathcal{F}^{+}(\xi,\lambda) - \mathcal{F}^{-}(\xi,\lambda) = -2i(b-a)\,\frac{e^{\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} .$$

We shall now prove

Theorem 4.1:

The formulae

$$f(\lambda) = \frac{1}{\pi}\sqrt{\frac{b-a}{2}}\ \underset{\delta\to 0,\delta>0}{\mathrm{l.i.m.}} \int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}}\, g(\xi)\,d\xi$$

$$g(\xi) = \frac{1}{\pi}\sqrt{\frac{b-a}{2}}\ \underset{\delta\to 0,\delta>0}{\mathrm{l.i.m.}} \int_{a+\delta}^{b-\delta} \frac{e^{-\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}}\, f(\lambda)\,d\lambda$$

are inversion formulae which represent an isometry between the spaces $L^2(a,b)$ and $L^2(-1,1)$.

Proof: The Fourier inversion formulae are

$$F(v) = \frac{1}{\sqrt{2\pi}}\ \underset{N,M\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{M} e^{iuv} G(u)\,du$$

$$G(u) = \frac{1}{\sqrt{2\pi}}\ \underset{N,M\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{M} e^{-iuv} F(v)\,dv . \qquad [5] \quad .$$

We now introduce the new variables

$$u = \frac{1}{\sqrt{2\pi}} \log\frac{1+\xi}{1-\xi} , \qquad v = \frac{1}{\sqrt{2\pi}} \log\frac{\lambda-a}{b-\lambda} ,$$

so that $\frac{du}{d\xi} = \sqrt{\frac{2}{\pi}}\,\frac{1}{1-\xi^2}$, $\frac{dv}{d\lambda} = \frac{b-a}{\sqrt{2\pi}}\,\frac{1}{(\lambda-a)(b-\lambda)}$. Next, we set $G(u) = \sqrt[4]{\frac{\pi}{2}}\,\sqrt{1-\xi^2(u)}\; G_1(u)$. Thus

$$\int_{-\infty}^{\infty} |G(u)|^2 du = \int_{-1}^{1} |G_1(u(\xi))|^2 d\xi \quad .$$

If $G(u) \varepsilon L^2(-\infty,\infty)$, then $F(v) \varepsilon L^2(-\infty,\infty)$. Furthermore,

$$\int_{-\infty}^{\infty} |F(v)|^2 dv = \int_{-\infty}^{\infty} |G(u)|^2 du \ . \quad [5] \ .$$

We now set $F(v) = \frac{\sqrt[4]{2\pi}}{\sqrt{b-a}} \sqrt{(\lambda(v)-a)(b-\lambda(v))}\ F_1(v)$. Then

$$\int_{-\infty}^{\infty} |F(v)|^2 dv = \int_{a}^{b} |F_1(v(\lambda))|^2 d\lambda \quad .$$

Furthermore,

$$F_1(v(\lambda)) = \frac{\sqrt{b-a}}{(2\pi)^{3/4}} \underset{\delta\rightarrow,\delta>0}{\text{l.i.m.}} \int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{(\frac{\pi}{2})^{1/4}\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} G_1(u(\xi))d\xi$$

$$= \frac{1}{\pi}\sqrt{\frac{b-a}{2}} \underset{\delta\rightarrow0,\delta>0}{\text{l.i.m.}} \int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} G_1(u(\xi))d\xi \quad .$$

Similarly,

$$G_1(u(\xi)) = \frac{1}{\pi}\sqrt{\frac{b-a}{2}} \underset{\delta\rightarrow0,\delta>0}{\text{l.i.m.}} \int_{a+\delta}^{b-\delta} \frac{e^{-\frac{i}{2\pi}(\log\frac{1+\xi}{1-\xi})(\log\frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} F_1(v(\lambda))d\lambda \ .$$

Next, we turn to

Lemma 4.1:

The finite Hilbert transform

$$Kx(\lambda) = \frac{1}{\pi i}\, P \int_{a}^{b} \frac{x(\mu)}{\mu-\lambda}\, d\mu$$

is a bounded operator on $L^2(a,b)$ with norm $||K|| \leq 1$.

Proof: The Hilbert transform

$$Hx(\lambda) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{x(\mu)}{\mu-\lambda} d\mu$$

is a unitary operator on $L^2(-\infty,\infty)$ [5] . If we now take $x(\lambda)$ $\varepsilon L^2(a,b)$ and define $x(\lambda) = 0$, $\lambda < a$, $\lambda > b$. then the resulting function belongs to $L^2(-\infty,\infty)$. We then have

$$\int_{-\infty}^{\infty} |(Hx(\lambda))(\mu)|^2 d\mu = \int_{-\infty}^{\infty} |x(\lambda)|^2 d\lambda = \int_{a}^{b} |x(\lambda)|^2 d\lambda .$$

Furthermore, for $a < \mu < b$, $(Hx(\lambda))(\mu) = (Kx(\lambda))(\mu)$. Of course,

$$\int_{a}^{b} |(Kx(\lambda))(\mu)|^2 d\mu \leq \int_{-\infty}^{\infty} |(Hx(\lambda))(\mu)|^2 d\mu = \int_{a}^{b} |x(\lambda)|^2 d\lambda \quad ,$$

so that $||Kx(\lambda)|| \leq ||x(\lambda)||$.

We are now in a position to prove

Theorem 4.2:

The inversion formulae of Theorem 4.1 give the spectral representation of the finite Hilbert transform, i.e. if $f(\lambda)$ corresponds to $g(\xi)$, then $Kf(\lambda)$ corresponds to $\xi g(\xi)$.

Proof: The set of Hölder continuous functions on the interval (-1,1), which vanish identically near the points -1 and 1 is everywhere dense in the space $L^2(-1,1)$. The set of images of these elements under the isometry of Theorem 4.1 is everywhere dense in

$L^2(a,b)$. Let $g(\xi)$ be a function on the interval $(-1,1)$ which belongs to the set described above. Its image then has the form

$$f(\lambda) = \frac{1}{\pi}\sqrt{\frac{b-a}{2}} \int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log \frac{1+\xi}{1-\xi})(\log \frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} g(\xi)\,d\xi$$

where δ is sufficiently small. We now apply the finite Hilbert transform K to $f(\lambda)$:

$$\frac{1}{\pi i} P\int_a^b \frac{f(\mu)}{\mu-\lambda}\,d\mu = \frac{1}{\pi^2 i}\sqrt{\frac{b-a}{2}}\, P\int_a^b \frac{1}{\mu-\lambda}\Big\{\int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log \frac{1+\xi}{1-\xi})(\log \frac{\mu-a}{b-\mu})}}{\sqrt{(1-\xi^2)(\mu-a)(b-\mu)}} g(\xi)\,d\xi\Big\}\,d\mu$$

$$= \frac{1}{\pi^2 i}\sqrt{\frac{b-a}{2}} \int_{-1+\delta}^{1-\delta} \Big\{P\int_a^b \frac{e^{\frac{i}{2\pi}(\log \frac{1+\xi}{1-\xi})(\log \frac{\mu-a}{b-\mu})}}{\sqrt{(1-\xi^2)(\mu-a)(b-\mu)}} \frac{d\mu}{\mu-\lambda}\Big\} g(\xi)\,d\xi\ , \qquad [6]$$

$$= \frac{1}{\pi}\sqrt{\frac{b-a}{2}} \int_{-1+\delta}^{1-\delta} \frac{e^{\frac{i}{2\pi}(\log \frac{1+\xi}{1-\xi})(\log \frac{\lambda-a}{b-\lambda})}}{\sqrt{(1-\xi^2)(\lambda-a)(b-\lambda)}} \xi g(\xi)\,d\xi\ ,$$

since our kernel is an eigenfunction for the operator K . Thus $Kf(\lambda)$ corresponds to $\xi g(\xi)$ when $g(\xi)$ belongs to our set. We now use the fact that our set is everywhere dense, the isometry of Theorem 4.1, and the boundedness of the operator K to extend this statement to all functions $g(\xi)$ of $L^2(-1,1)$.

Corollary:

The finite Hilbert transform K has a simple continuous spectrum which consists of all points ξ with $-1 \leq \xi \leq 1$. Thus $||K|| = 1$.

5. Formal Aspects.

We now return to the general discussion. In order to obtain a spectral representation of (1.1), we would like to use an isometric transformation which is, if possible, associated with an integral transformation whose kernel is related to the generalized eigenfunctions of Section 2. We thus proceed in the following way:

Suppose that $g(\xi)$ belongs to the space $L^2(a-\varepsilon,b+\varepsilon)$. We then define the analytic function $X(z)$ by the formula

$$(5.1)\qquad X(z) = \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,z,\varepsilon)}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)}\, g(\xi)\,d\xi \quad ,$$

where we use the definition (2.11). Since $X(\infty) = 0$, we then hope that there exists a function $f(\lambda)$ such that

$$(5.2)\qquad X(z) = \frac{1}{2\pi i}\int_a^b \frac{f(\mu)}{\mu-z}\,d\mu$$

Thus, formally, it is plausible to define the following limiting process:

$$(5.3)\qquad f(\lambda) = X^+(\lambda) - X^-(\lambda)$$

$$= \frac{1}{2\pi i}\lim_{\eta\to 0,\ \eta>0}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,\lambda+i\eta,\varepsilon)-F(\xi,\lambda-i\eta,\varepsilon)}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)}\, g(\xi)\,d\xi \ .$$

Observe that if it were possible to interchange the limiting and integration processes in the last expression, then we would obtain an integral transform whose kernel is an improper eigenfunction of

the operator L_ε . Similarly, we define the analytic function $Y(\ell)$ by the formula

$$(5.4) \qquad Y(\ell) = \frac{1}{2\pi i} \int_a^b \frac{H(\ell,\lambda,-\varepsilon)}{2\varepsilon} f(\lambda)\,d\lambda$$

and, we shall consider the transformation generated by $Y^+(\xi) - Y^-(\xi)$, i.e.

$$(5.5) \qquad \frac{1}{2\pi i} \lim_{\eta\to 0,\,\eta>0} \int_a^b \frac{H(\xi+i\eta,\lambda,-\varepsilon)-H(\xi-i\eta,\lambda,-\varepsilon)}{2\varepsilon} f(\lambda)\,d\lambda \ .$$

(For the definition of $H(\ell,\lambda,\varepsilon)$, see (3.6).)

We shall now show that, formally, (5.3) and (5.5) constitute a pair of inversion formulae. We first "prove" that

$$(5.6) \quad g(\xi) = \frac{1}{2\pi i} \lim_{\nu\to 0} \int_a^b \frac{H(\xi+i\nu,\lambda,-\varepsilon)-H(\xi-i\nu,\lambda,-\varepsilon)}{2\varepsilon}$$

$$\cdot\Big\{ \frac{1}{2\pi i} \lim_{\eta\to 0} \int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\gamma,\lambda+i\eta,\varepsilon)-F(\gamma,\lambda-i\eta,\varepsilon)}{\max(\gamma-\varepsilon,a)-\min(\gamma+\varepsilon,b)} g(\gamma)\,d\gamma \Big\}\,d\lambda \ .$$

Let us examine

$$\frac{1}{2\pi i}\int_a^b \frac{H(\ell,\lambda,-\varepsilon)}{2\varepsilon}\Big\{\frac{1}{2\pi i} \lim_{\eta\to 0} \int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\gamma,\lambda+i\eta,\varepsilon)-F(\gamma,\lambda-i\eta,\varepsilon)}{\max(\gamma-\varepsilon,a)-\min(\gamma+\varepsilon,b)} g(\gamma)\,d\gamma\Big\}\,d\lambda \ .$$

Formally, we may write the last expression in the form

$$\frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \Big\{\lim_{\eta\to 0} \frac{1}{2\pi i} \int_a^b \frac{H(\ell,\lambda,-\varepsilon)\{H(\gamma+i\eta,\lambda,\varepsilon)-H(\gamma-i\eta,\lambda,\varepsilon)\}}{2\varepsilon\{\max(\gamma-\varepsilon,a)-\min(\gamma+\varepsilon,b)\}}\,d\lambda\Big\}\, g(\gamma)\,d\gamma \ ,$$

since $H_+(\xi,\lambda,\varepsilon) - H_-(\xi,\lambda,\varepsilon) = F^+(\xi,\lambda,\varepsilon) - F^-(\xi,\lambda,\varepsilon)$.

Now we observe that (see (3.3))

$$H(\zeta,\lambda,-\varepsilon) = -\overline{H(\bar{\zeta},\lambda,\varepsilon)} \quad . \tag{5.7}$$

If we utilize (3.11), we see that our last integral expression is simply equal to

$$\frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{g(\gamma)}{\gamma-\zeta}\, d\gamma \quad .$$

This, however, "establishes" (5.6) .

We also wish to "prove" that

(5.8) $f(\lambda) =$

$$\frac{1}{2\pi i}\lim_{\nu\to 0}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,\lambda+i\nu,\varepsilon)-F(\xi,\lambda-i\nu,\varepsilon)}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)}\Big\{\frac{1}{2\pi i}\lim_{\eta\to 0}\int_a^b \frac{H(\xi+i\eta,\omega,-\varepsilon)-H(\xi-i\eta,\omega,-\varepsilon)}{2\varepsilon}$$

$$\cdot\, f(\omega)d\omega\Big\}d\xi \quad .$$

Let us examine

$$\frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,z,\varepsilon)}{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)}\Big\{\frac{1}{2\pi i}\lim_{\eta\to 0}\int_a^b \frac{H(\xi+i\eta,\omega,-\varepsilon)-H(\xi-i\eta,\omega,-\varepsilon)}{2\varepsilon}f(\omega)d\omega\Big\}$$

This is formally equal to

$$\frac{1}{2\pi i}\int_a^b \{\lim_{\eta\to 0} \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,z,\varepsilon)\{F(\xi,\omega+i\eta,-\varepsilon)-F(\xi,\omega-i\eta,-\varepsilon)\}}{2\varepsilon\{\max(\xi-\varepsilon,a)-\min(\xi+\varepsilon,b)\}}\, d\xi\} f(\omega)\,d\omega \ .$$

Of course,

$$F(\xi,z,-\varepsilon) = \overline{-F(\xi,\overline{z},\varepsilon)} \ . \tag{5.9}$$

Utilizing (5.9) and (3.16), we obtain

$$\frac{1}{2\pi i}\int_a^b \frac{f(\omega)}{\omega-z}\, d\omega$$

for our last integral expression. This "establishes" (5.8).

We next turn to the formulae

$$g(\xi) = \frac{1}{2\pi i}\lim_{\eta\to ,\, \eta>0} \int_a^b \frac{H(\xi+i\eta,\lambda,-\varepsilon)-H(\xi-i\eta,\lambda,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\, f(\lambda)\,d\lambda \tag{5.10}$$

and

$$f(\lambda) = -\frac{1}{2\pi i}\lim_{\eta\to 0,\, \eta>0} \int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,\lambda+i\eta,\varepsilon)-F(\xi,\lambda-i\eta,\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\, g(\xi)\,d\xi \ . \tag{5.11}$$

The "proof" above shows that (5.10) and (5.11) are a pair of "inversion formulae". We now claim that, formally, formulae (5.10) and (5.11) represent an isometry between the spaces $L^2(a,b)$ and $L^2(a-\varepsilon,b+\varepsilon)$. (Observe that $H_+(\xi,\lambda,-\varepsilon) - H_-(\xi,\lambda,-\varepsilon) = \overline{H_+(\xi,\lambda,\varepsilon)} - \overline{H_-(\xi,\lambda,\varepsilon)} = \overline{F^+(\xi,\lambda,\varepsilon) - F^-(\xi,\lambda,\varepsilon)}$, so that the "kernels" of

(5.10) and (5.11) are complex conjugates.) Formally,

$$\int_{a-\varepsilon}^{b+\varepsilon} |g(\xi)|^2 d\xi = \int_{a-\varepsilon}^{b+\varepsilon} g(\xi)\overline{g(\xi)}d\xi$$

$$= \int_{a-\varepsilon}^{b+\varepsilon} \{\frac{1}{2\pi i} \lim_{\eta \to 0} \int_a^b \frac{H(\xi+i\eta,\lambda,-\varepsilon)-H(\xi-i\eta,\lambda,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}} f(\lambda)d\lambda$$

$$\cdot - \frac{1}{2\pi i} \lim_{\nu \to 0} \int_a^b \frac{H(\xi+i\nu,\mu,\varepsilon)-H(\xi-i\nu,\mu,\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}} \overline{f(\mu)}d\mu\} \; d\xi$$

$$= \frac{1}{4\pi^2}\int_a^b f(\lambda)\{\lim_{\substack{\eta \to 0 \\ \nu \to 0}} \int_a^b$$

$$\cdot\{\int_{a-\varepsilon}^{b+\varepsilon} \frac{\{H(\xi+i\eta,\lambda,-\varepsilon)-H(\xi-i\eta,\lambda-\varepsilon)\}\{H(\xi+i\nu,\mu,\varepsilon)-H(\xi-i\nu,\mu,\varepsilon)\}}{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}} d\xi \overline{f(\mu)}d\mu\}d\lambda$$

$$= \frac{1}{4\pi^2}\int_a^b f(\lambda)\{\lim_{\substack{\eta \to 0 \\ \nu \to 0}} \int_a^b$$

$$\cdot\{\int_{a-\varepsilon}^{b+\varepsilon} \frac{\{F(\xi,\lambda+i\eta,-\varepsilon)-F(\xi,\lambda-i\eta,-\varepsilon)\}\{F(\xi,\mu+i\nu,\varepsilon)-F(\xi,\mu-i\nu,\varepsilon)\}}{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}} d\xi \overline{f(\mu)}d\mu\}d\lambda$$

$$= \int_a^b f(\lambda)\overline{f(\lambda)}d\lambda \; .$$

6. A Rigorous Proof of the Spectral Representation for the Operator (1.1).

Let ω_1, ω_2, ω_2, ..., ω_n ... be a strictly monotone, unbounded sequence of positive numbers. G. Szegö [1, p. 101] has shown that the sequence $\{\frac{1}{\lambda+\omega_n}\}$ is complete in the space $L^2(a,b)$. Of course $\{\frac{1}{\xi+\omega_n}\}$ is complete in $L^2(a-\varepsilon,b+\varepsilon)$. (We shall always

assume that ω_1 is so large that it lies outside the two intervals which are involved.)

We now denote the transformations (5.10) and (5.11) by $Uf(\lambda)$ and $Vg(\xi)$ respectively.

Let us compute $U\{\frac{1}{\lambda+\omega_n}\}$. By the Cauchy Residue Theorem

$$(6.1)\quad \frac{1}{2\pi i}\oint E(\ell,z,-\varepsilon)\,\frac{1}{z+\omega_n}\,dz = E(\ell,-\omega_n,-\varepsilon) - 1$$

so that

$$(6.2)\quad \frac{1}{2\pi i}\int_a^b H(\ell,\lambda,-\varepsilon)\,\frac{1}{\lambda+\omega_n}\,d\lambda = E(\ell,-\omega_n,-\varepsilon) - 1 .$$

Hence

$$(6.3)\quad U\{\frac{1}{\lambda+\omega_n}\} = \frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}} .$$

Next, we compute $V\{\frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\}$. From (3.15) we obtain

$$-\frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon}\frac{F(\xi,z,\varepsilon)F(\xi,-\omega_n,-\varepsilon)}{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}\,d\xi$$

$$= \frac{1}{2\pi i}\,\frac{1}{z+\omega_n}\{\int_a^b\frac{d\mu}{\mu-z} - \int_a^b\frac{d\mu}{\mu+\omega_n}\} .$$

Hence

$$(6.4)\qquad V\{\frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\} = \frac{1}{\lambda+\omega_n} .$$

We also compute $V\{-\frac{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\}$. By the Cauchy Residue Theorem

$$(6.5)\quad \frac{1}{2\pi i}\oint E(\ell,z,\varepsilon)\,\frac{1}{\ell+\omega_n}\,d\ell = E(-\omega_n,z,\varepsilon) - 1$$

so that

$$(6.6)\quad \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon} F(\xi,z,\varepsilon)\,\frac{1}{\xi+\omega_n}\,d\xi = E(-\omega_n,z,\varepsilon) - 1 \,.$$

Hence

$$(6.7)\quad V\{-\frac{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\} = H(-\omega_n,\lambda,\varepsilon) \,.$$

Furthermore, from (3.10) we have

$$\frac{1}{2\pi i}\int_a^b H(\ell,\lambda,-\varepsilon)H(-\omega_n,\lambda,\varepsilon)\,d\lambda = \frac{1}{2\pi i}\,\frac{2\varepsilon}{\ell+\omega_n}\int_{a-\varepsilon}^{b+\varepsilon}\{\max(\nu-\varepsilon,a)-\min(\nu+\varepsilon,b)\}$$
$$\cdot\,\{\frac{1}{\nu-\ell} - \frac{1}{\nu+\omega_n}\}\,d\nu$$

so that

$$(6.8)\quad U\{H(-\omega_n,\lambda,\varepsilon)\} = -\frac{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n} \,.$$

We now compute $U\{\sqrt{2\varepsilon}\}$. By the Cauchy Residue Theorem,

$$(6.9)\quad \frac{1}{2\pi i}\oint E(\ell,z,-\varepsilon)\,dz = \frac{1}{2\pi i}\int_a^b \log\frac{\mu-\ell+\varepsilon}{\mu-\ell-\varepsilon}\,d\mu$$

$$= \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon}\frac{\min(\nu+\varepsilon,b)-\max(\nu-\varepsilon,a)}{\nu-\ell}\,d\nu = \frac{1}{2\pi i}\int_a^b H(\ell,\lambda,-\varepsilon)\,d\lambda\ .$$

(use integration by parts)

Hence

$$(6.10)\qquad U\{\sqrt{2\varepsilon}\ \} = \sqrt{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)}\ .$$

Furthermore,

$$(6.11)\qquad \frac{1}{2\pi i}\oint E(\ell,z,\varepsilon)\,d\ell = -\frac{2\varepsilon}{2\pi i}\int_a^b\frac{d\mu}{\mu-z} = \frac{1}{2\pi i}\int_{a-\varepsilon}^{b+\varepsilon}F(\xi,z,\varepsilon)\,d\xi\ ,$$

so that

$$(6.12)\qquad V\{\sqrt{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)}\} = \sqrt{2\varepsilon}\quad .$$

One may easily verify that

$$(6.13)\qquad 2\varepsilon(b-a) = \int_{a-\varepsilon}^{b+\varepsilon}\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}\,d\xi\ .$$

From (3.10) we see that

$$(6.14)\quad \int_a^b\overline{H(-\omega_n,\lambda,\varepsilon)}\,H(-\omega_m,\lambda,\varepsilon)\,d\lambda = \int_{a-\varepsilon}^{b+\varepsilon}\frac{2\varepsilon\{\min(\nu+\varepsilon,b)-\max(\nu-\varepsilon,a)\}}{(\nu+\omega_n)(\nu+\omega_m)}\,d\nu\ .$$

From (3.15) we obtain

$$(6.15)\quad \int_{a-\varepsilon}^{b+\varepsilon}\frac{F(\xi,-\omega_n,-\varepsilon)\overline{F(\xi,-\omega_m,-\varepsilon)}}{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}\,d\xi = \int_a^b\frac{d\mu}{(\mu+\omega_n)(\mu+\omega_m)}\ .\quad \text{(use (5.9))}$$

From (6.2) and (6.6), we see that

$$(6.16)\quad \int_a^b \overline{H(-\omega_m,\lambda,\varepsilon)}\,\frac{1}{\lambda+\omega_n}\,d\lambda = -\int_{a-\varepsilon}^{b+\varepsilon} F(\xi,-\omega_n,-\varepsilon)\,\frac{1}{\xi+\omega_m}\,d\xi$$

$$= \int_{a-\varepsilon}^{b+\varepsilon} \left\{\frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\right\}$$

$$\cdot \left\{\frac{-\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_m}\right\}d\xi\,.$$

From (6.9) we have

$$(6.17)\quad \int_a^b H(-\omega_n,\lambda,\varepsilon)\,d\lambda = \int_{a-\varepsilon}^{b+\varepsilon} \frac{\max(\nu-\varepsilon,a)-\min(\nu+\varepsilon,b)}{\nu+\omega_n}\,d\nu$$

so that

$$(6.18)\quad \int_a^b H(-\omega_n,\lambda,\varepsilon)\cdot\sqrt{2\varepsilon}\,d\lambda$$

$$= \int_{a-\varepsilon}^{b+\varepsilon} -\frac{\sqrt{2\varepsilon\{\min(\nu+\varepsilon,b)-\max(\nu-\varepsilon,a)\}}}{\nu+\omega_n}\cdot\sqrt{\min(\nu+\varepsilon,b)-\max(\nu-\varepsilon,a)}\;d\nu\,.$$

From (6.11) we obtain

$$(6.19)\quad \int_{a-\varepsilon}^{b+\varepsilon} F(\xi,-\omega_n,-\varepsilon)\,d\xi = 2\varepsilon\int_a^b \frac{d\mu}{\mu+\omega_n}$$

so that

$$(6.20)\qquad \int_{a-\varepsilon}^{b+\varepsilon} \frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\cdot\sqrt{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)}\,d\xi = \int_a^b \frac{\sqrt{2\varepsilon}}{\mu+\omega_n}\,d\mu \quad .$$

Now let D_1 be the linear subset of $L^2(a,b)$, which is generated by the elements

$$\left\{ \frac{1}{\lambda+\omega_n} \;,\; H(-\omega_m,\lambda,\varepsilon) \;,\; \sqrt{2\varepsilon} \right\}, \qquad n,\, m = 1,2,3,\; \ldots \;.$$

Similarly, let D_2 be the linear subset of $L^2(a-\varepsilon,b+\varepsilon)$, which is generated by the elements $\left\{\frac{F(\xi,-\omega_n,-\varepsilon)}{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}\right.$, $-\frac{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_m}$, $\left.\sqrt{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)}\right\}$, $n, m = 1,2,3,\ldots$. Because of the remarks at the beginning of this section, one can easily show that D_1 and D_2 are everywhere dense in their respective Hilbert spaces. (6.3), (6.4), (6.7), (6.8), (6.10), (6.12) show that relative to D_1 and D_2

$$(6.21)\qquad UV = 1 \;,\quad VU = 1 \quad .$$

(6.13), (6.14), (6.15), (6.16), (6.18), (6.20) show that relative to D_1 and D_2, the transformations U and V are isometries, i.e.

$$(6.22)\qquad \|Uf(\lambda)\| = \|f(\lambda)\| \;,\quad f(\lambda)\ \varepsilon\ D_1$$

$$\|Vg(\xi)\| = \|g(\xi)\| \;,\quad g(\xi)\ \varepsilon\ D_2 \quad .$$

Thus if we restrict the domains of the operators U and V to the linear subsets D_1 and D_2 respectively, the operators are bounded and have unique bounded extensions $\tilde{U}$, $\tilde{V}$ defined for all elements of $L^2(a,b)$ and $L^2(a-\varepsilon,b+\varepsilon)$ respectively. Obviously

$$\tilde{U}\tilde{V} = 1 \ , \quad \tilde{V}\tilde{U} = 1 \tag{6.23}$$

and

$$||\tilde{U}f(\lambda)|| = ||f(\lambda)|| \ , \ ||\tilde{V}g(\xi)|| = ||g(\xi)|| \ . \tag{6.24}$$

Hence $\tilde{U}$ and $\tilde{V}$ are isometric mappings.

We shall now establish that if

$$f(\lambda) = \tilde{V}g(\xi) \ ,$$

then

$$L_\varepsilon f(\lambda) = \tilde{V}\xi g(\xi) \ .$$

We first remark that since $\tilde{V}$ is an isometry and $\{-\frac{\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\}$, $n = 1,2,3, \ldots$ is a complete set in $L^2(a-\varepsilon,b+\varepsilon)$, $\{H(-\omega_n,\lambda,\varepsilon)\}$, $n = 1,2,3, \ldots$ is a complete set in $L^2(a,b)$. We now compute $\tilde{V}\{\frac{\xi\{-\sqrt{2\varepsilon \ \min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\}$.

Observe that $\frac{\xi}{\xi+\omega_n} = \frac{-\omega_n}{\xi+\omega_n} + 1$ so that the element whose image we shall

compute, lies in D_2 . In D_2, howe er, $\tilde{V} = V$. Using (6.7) and (6.12), we obtain

$$(6.25) \qquad \tilde{V}\left\{\frac{\xi\{-\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\right\} = -\omega_n H(-\omega_n,\lambda,\varepsilon) - 2\varepsilon$$

However, from (3.1) and (3.6) we see that

$$(6.26) \quad -\omega_n H(-\omega_n,\lambda,\varepsilon) - 2\varepsilon = \lambda H(-\omega_n,\lambda,\varepsilon) + \varepsilon\{[E^+(-\omega_n,\lambda,\varepsilon)-1] + [E^-(-\omega_n,\lambda,\varepsilon)-1]\} \ .$$

But $E(-\omega_n, z, \varepsilon) - 1 = \frac{1}{2\pi i}\int_a^b \frac{H(-\omega_n,\mu,\varepsilon)}{\mu-z}\,d\mu$, so that the last expression is equal to

$$\lambda H(-\omega_n,\lambda,\varepsilon) + \frac{\varepsilon}{\pi i}\, P\int_a^b \frac{H(-\omega_n,\mu,\varepsilon)}{\mu-\lambda}\,d\mu = L_\varepsilon\{H(-\omega_n,\lambda,\varepsilon)\} \ .$$

Thus we have established that

$$(6.27) \quad L_\varepsilon\{H(-\omega_n,\lambda,\varepsilon)\} = \tilde{V}\left\{\xi\left\{\frac{-\sqrt{2\varepsilon\{\min(\xi+\varepsilon,b)-\max(\xi-\varepsilon,a)\}}}{\xi+\omega_n}\right\}\right. \ .$$

Because L_ε is a bounded operator and because the sets of functions involved are complete, we may immediately conclude that if

$$f(\lambda) = \tilde{V}g(\xi) \quad ,$$

then

$$L_\varepsilon f(\lambda) = \tilde{V}\xi g(\xi) \qquad .$$

This establishes a spectral representation for the operator L_ε , so that we have

Theorem 6.1:

The operator L_ε (see (1.1)) has a simple continuous spectrum which consists of all points ξ with $a-\varepsilon \leq \xi \leq b+\varepsilon$.

Footnotes

[1] The perturbation $L + \varepsilon \cdot 1$, for example, causes a shift in a trivial manner.

[2] The sequence may possibly contain only a finite number of distinct elements.

[3] We cut the ℓ-plane along the real axis from -1 to 1.

Bibliography

[1] Courant, R. and Hilbert D., "Methods of Mathematical Physics," Interscience, New York 1953.

[2] Michlin, S. G., "Singular Integral Equations," American Math. Soc., Translation No. 24, 1950.

[3] Muskhelishvili, N. I., "Singular Integral Equations," (translated from the Russian), Noordhoff, Groningen 1953.

[4] Riesz, F. and Nagy, B., "Lecons d'Analyse Fonctionnelle, Budapest 1953.

[5] Titchmarsh, E. C., "Introduction to the Theory of Fourier Integrals," Oxford 1937.

[6] Tricomi, F. G., "Sull'inversione dell'ordine di integrali 'principali' nel senso di Cauchy," Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8), 18, 1955.

Date Due

[illegible]
XEROX
FEB 17 1970
FEB 19 1970
MAR 5 1970
MAR 5 1970

PRINTED IN U. S. A.

NYU
IMM
242

c.1

Koppelman
Perturbation of continuous

NYU c. 1

NYU
IMM
242 c. 1

Koppelman
AUTHOR

Perturbation of continuous spectra and...
TITLE

DATE DUE	BORROWER'S NAME	ROOM NUMBER
MAR 19 '[illegible]	~~E. Theil~~	
FEB [illegible] '7[illegible]	~~L. Bradley~~	
FEB 19 197[illegible]	~~L. R. Bradley~~	
MAR 5 1978	[illegible]	

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.